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AN INTEGRAL FORMULA FOR IMMERSIONS IN EUCLIDEAN SPACE

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1. Introduction

This paper derives a general rigidity theorem and an integral formula for immersions of a compact oriented riemannian manifold without boundary in a euclidean space. The formula is applied to a volume-preserving immersion to establish a simple geometric criterion that the immersion be isometric. As the integral formula has a formal resemblance to one derived by Chern and Hsiung in [1], we conclude the paper with some remarks about that work.

2. Notations and conventions

Let M be a compact oriented m-dimensional riemannian manifold without boundary with metric $ds^{2\#}$, and let

$$X: M \to R^{m+n}$$

be an immersion in an $(m+n)$-dimensional euclidean space R^{m+n}. As such M admits a second riemannian metric,

$$ds^2 = dX \cdot dX \ .$$

We fix the range of indices so that the capital Latin indices run from 1 to $m+n$, the small Greek indices from 1 to m, and the small Latin indices from $m+1$ to $m+n$.

Matters being so, we choose orthonormal coframes $\{\tau^{\alpha\#}\}$ for $ds^{2\#}$ on M which diagonalize ds^2 with respect to $ds^{2\#}$. Thus

$$ds^{2\#} = \Sigma(\tau^{\alpha\#})^2 \ , \qquad ds^2 = \Sigma g_\alpha(\tau^{\alpha\#})^2 \ ,$$

and the first invariants of the pair of metrics are the elementary symmetric functions in the functions g_α.

Next we choose a family of orthonormal frames $\{e_A\}$ on $X(M)$ in R^{m+n} in such a way that $\{e_\alpha\}$ are unit tangent vectors of $X(M)$ and the pull back of the dual coframe $\{\tau^A\}$ satisfies

$$\tau^\alpha = h_\alpha \tau^{\alpha\#} \ ,$$

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where $h_\alpha = (g_\alpha)^{1/2}$. As such the volume elements of ds^2 and $ds^{2\#}$ are respectively

$$dV = \tau^1 \wedge \cdots \wedge \tau^m , \qquad dV^\# = \tau^{1\#} \wedge \cdots \wedge \tau^{m\#} .$$

The pull back of the structure equations

$$de_A = \Sigma \varphi_A^B e_B ,$$
$$d\tau^B = \Sigma \tau^A \wedge \varphi_A^B ,$$
$$d\varphi_A^B = \Sigma \varphi_A^C \wedge \varphi_C^B$$

of R^{m+n} give rise to a skew-symmetric matrix of linear differential forms

$$\varphi_\alpha^\beta = \Sigma \Gamma_{\alpha\gamma}^\beta \tau^\gamma ,$$

called the Levi-Civita connection for ds^2, and a vector of quadratic differential forms

$$\Sigma \tau^\alpha \odot \varphi_\alpha^a = \Sigma A_{\alpha\beta}^a \tau^\alpha \odot \tau^\beta ,$$

called the vector-valued second fundamental form.

The exterior differential equations

$$d\tau^{\alpha\#} = \Sigma \tau^{\alpha\#} \wedge \varphi_\gamma^{\alpha\#} ,$$
$$\varphi_\gamma^{\alpha\#} = -\varphi_\alpha^{\gamma\#}$$

define a unique skew-symmetric matrix of linear differential forms

$$\varphi_\alpha^{\beta\#} = \Sigma \Gamma_{\alpha\gamma}^{\beta\#} \tau^\gamma ,$$

called the Levi-Civita connection for $ds^{2\#}$. This matrix allows us to introduce a covariant differentiation with respect to $ds^{2\#}$. Thus, if f is a function we introduce $f_{;\alpha}$ by

$$df = \Sigma f_{;\alpha} \tau^{\alpha\#} ;$$

if $w = \Sigma a_\alpha \tau^{\alpha\#}$ is a linear differential form then we introduce $a_{\alpha;\beta}$ by

$$da_\alpha - \Sigma a_\gamma \varphi_\alpha^{\gamma\#} = \Sigma a_{\alpha;\beta} \tau^{\beta\#} ;$$

if $Q = \Sigma b_{\alpha\beta} \tau_\alpha^{\alpha\#} \odot \tau^{\beta\#}$ is a quadratic differential form then we introduce $b_{\alpha\beta;\gamma}$ by

$$db_{\alpha\beta} - \Sigma \varphi_\alpha^{\gamma\#} b_{\gamma\beta} - \Sigma b_{\alpha\gamma} \varphi_\beta^{\gamma\#}$$
$$= \Sigma b_{\alpha\beta;\gamma} \tau^{\gamma\#} .$$

Finally we introduce the Hodge mapping defined with respect to $ds^{2\#}$, which is the linear mapping $*_\#$ characterized by

$$*_\#(\tau^{\alpha\#}) = (-1)^{\alpha-1} \tau^{1\#} \wedge \cdots \wedge \tau^{\alpha-1\#} \wedge \tau^{\alpha+1\#} \wedge \cdots \wedge \tau^{m\#} .$$

As such if $w = \Sigma a_\alpha \tau^{\alpha\#}$ is a linear differential form then $d *_\# w$ is an exact m-form, and a short calculation proves that

$$d *_\# w = \Sigma a_{\alpha;\alpha} \tau^{1\#} \wedge \cdots \wedge \tau^{m\#} = \Sigma a_{\alpha;\alpha} dV^\# .$$

We recall that if $w = df$, where f is a real-valued function, then

$$d *_{\sharp} df = \Delta_{\sharp}(f)dV ,$$

where $\Delta_{\sharp}(f)$ is the Laplacian of f taken with respect to the metric $ds^{2\sharp}$.

These operations make sense in the case that $ds^{2\sharp} = ds^2$, and we will denote the Laplacian with respect to ds^2 by Δ.

3. The integral formula

Let 0 denote a choice of origin in R^{m+n}; then the linear differential form

$$\Omega = \Sigma(X \cdot e_\alpha)\tau^\alpha = \tfrac{1}{2}X \cdot dX$$

is defined independent of the particular family of the orthonormal frames $\{e_\alpha\}$ and orthonormal coframes $\{\tau^\alpha\}$, and hence induces a globally defined differential form on M. As such Stokes' theorem applies to yield the integral formula

$$0 = \int_M d *_{\sharp} \Omega = \int_M \Delta_{\sharp}(\tfrac{1}{2}X \cdot X)dv . \tag{3.1}$$

The explicit expression of the resulting integral formula is simplified by the introduction of the vector

$$\begin{aligned} h^* = {} & \Sigma A^a_{\alpha\alpha} h^2_\alpha e_a + \Sigma(\Gamma^\beta_{\alpha\alpha} - \Gamma^{\beta\sharp}_{\alpha\alpha}) h^2_\alpha e_\beta \\ & + \Sigma(h_\alpha \delta^\beta_\alpha)_{;\beta} e_\beta . \end{aligned} \tag{3.2}$$

The naturality of this vector is apparent from the following proposition.

Proposition 3.3. *Let a be any fixed vector in R^{m+n}; then*

$$\Delta_{\sharp}(a \cdot X) = a \cdot h^* . \tag{3.3}$$

Proof. Utilizing the structure equations, we have

$$\begin{aligned} d(a \cdot X) &= \Sigma(a \cdot e_\alpha) h_\alpha \tau^{\alpha\sharp} , \\ d(a \cdot e_\alpha) h_\alpha &- \Sigma\varphi^{\beta\sharp}_\alpha (a \cdot e_\beta) h_\beta \\ &= \Sigma(a \cdot e_i) A^i_{\alpha\gamma} h_\alpha h_\gamma \tau^{\gamma\sharp} + \Sigma(a \cdot e_\beta)(\Gamma^\beta_{\alpha\gamma} - \Gamma^{\beta\sharp}_{\alpha\gamma}) h_\alpha h_\gamma \tau^{\gamma\sharp} \\ &\quad + \Sigma(a \cdot e_\beta) h_\gamma h_\alpha \Gamma^{\beta\sharp}_{\alpha\gamma}) \tau^{\gamma\sharp} , \end{aligned}$$

and hence contracting the coefficients on α and γ gives (3.3) as claimed.

In particular this last Proposition is true if $ds^{2\sharp} = ds^2$. In this case the vector characterized by the last proposition will be denoted by h. We note that

$$h = \Sigma A^i_{\alpha\alpha} e_i , \tag{3.4}$$

which is the mean curvature vector of the immersion.

With this preparation the integral formula obtained from (3.1) may be stated as follows.

Theorem 3.4. *Let M be a compact oriented manifold without boundary endowed with the riemannian metric $ds^{2\#} = \Sigma(\tau^{\alpha\#})^2$, and let*

$$X: M \to R^{m+n}$$

be an immersion with induced metric $ds^2 = \Sigma g_\alpha(\tau^{\alpha\#})^2$, then

$$0 = \int_M (\Sigma g_\alpha + X\cdot h^*)dV^\# . \tag{3.5}$$

Proof. Since

$$\begin{aligned} d(X\cdot e_\alpha)h_\alpha - (X\cdot e_\gamma)h_\gamma\varphi_\alpha^{\gamma\#} &= \tau^\alpha h_\alpha + \Sigma(X\cdot e_\gamma)\varphi_\alpha^\gamma h_\alpha + \Sigma(X\cdot e_i)\varphi_\alpha^i h_\alpha \\ &\quad + (X\cdot e_\alpha)dh_\alpha - \Sigma(X\cdot e_\gamma)h_\gamma\varphi_\alpha^{\gamma\#} \\ &= g_\alpha\tau^{\alpha\#} + \Sigma(X\cdot e_\gamma)(\varphi_\alpha^\gamma - \varphi_\alpha^{\gamma\#})h_\alpha \\ &\quad + \Sigma(X\cdot e_\gamma)(dh_\alpha\delta_\alpha^\gamma - h_\gamma\varphi_\alpha^{\gamma\#})h_\alpha \\ &\quad + \Sigma(X\cdot e_i)\varphi_\alpha^i h_\alpha , \end{aligned}$$

we have

$$\begin{aligned} (\Sigma(X\cdot e_\alpha)h_\alpha)_{;\alpha} &= \Sigma g_\alpha + \Sigma(X\cdot e_\alpha)(\Gamma_{\gamma\gamma}^\alpha - \Gamma_{\gamma\gamma}^{\alpha\#})g_\gamma \\ &\quad + \Sigma(X\cdot e_\gamma)(h_\alpha\delta_\alpha^\gamma)_{;\alpha} + \Sigma(X\cdot e_i)A_{\alpha\alpha}^i g_\alpha \\ &= \Sigma g_\alpha + X\cdot h^* , \end{aligned}$$

which gives (3.5) by integration.

We note that applying the formula to the special case, where $ds^{2\#} = ds^2$, gives

$$0 = \int_M (m + X\cdot h)dV , \tag{3.6}$$

which is a classical formula of Minkowski.

4. Applications to volume-preserving immersions

Theorem 4.1. *Let $X: M \to R^{m+n}$ be an immersion of a compact oriented riemannian manifold without boundary. Then among all volume-preserving diffeomorphisms, the isometries are characterized as those for which the integral*

$$-\int_M X\cdot h^* dV$$

attains the minimal value of m times the value of vol. M.

Proof. By Newton's inequality, the hypothesis of volume-preserving implies

$$\frac{1}{m}\Sigma g_\alpha \geq (\Pi g_\alpha)^{1/m} = 1 \ ,$$

or

$$\Sigma g_\alpha - m \geq 0 \tag{4.2}$$

with equality if and only if

$$g_\alpha = 1 \qquad (1 \leq \alpha \leq m) \ . \tag{4.3}$$

As such substraction of (3.5) from (3.6), together with the hypothesis that $dV^\# = dV$, gives

$$0 = \int_M [(\Sigma g_\alpha - m) + X \cdot (h^* - h)] dV \ ,$$

but then (4.2) implies

$$\int_M X \cdot (h^* - h) dV \leq 0 \ ,$$

or

$$\int_M X \cdot h^* dV^\# \leq \int_M X \cdot h \, dV = -m \operatorname{vol} M \ .$$

If this maximum is achieved, then the integral formula becomes

$$0 = \int_M (\Sigma g_\alpha - m) dV \ ,$$

and hence (4.2) forces

$$\Sigma g_\alpha - m = 0 \ ,$$

and the equality statement (4.3) implies that the immersion is an isometry.

Corollary 4.4. *Let $X: M \to R^{m+n}$ be a volume-preserving immersion of a compact oriented riemannian manifold without boundary. Then*

$$h^* = h$$

if and only if the immersion is isometric.

5. A general rigidity theorem

Now consider the situation that the metric $ds^{2\#}$ comes from a second immersion. Thus we have the picture

$$\begin{array}{ccc} M & \xrightarrow{X} & R^{m+n} \\ & \searrow^{X^\#} & \\ & & R^{m+n} \end{array}$$

with $ds^2 = dX \cdot dX$ and $ds^{2\#} = dx^\# \cdot dx^\#$.

Theorem 5. *A necessary and sufficient condition that two immersions of a compact oriented manifold without boundary differ by a translation is that*

$$h^* = h_\# ,$$

where h^* is defined by (3.2), *and $h_\#$ is the mean curvature vector of the $X^\#$ immersion.*

Proof. By Proposition 3.3 we have

$$\Delta_\#(X - X^\#)\cdot a = (h^* - h_\#)\cdot a .$$

Therefore $X - X^\# =$ constant if and only if $h^* = h_\#$.

As a corollary we obtain the rigidity theorem that two isometric immersions of a compact oriented riemannian manifold without boundary differ by a translation if and only if they have the same mean curvature vectors. In the case of hypersurfaces this was a problem proposed by Minkowski.

6. Remarks on the paper of Chern and Hsiung

The integral formula in [1] was derived for volume-preserving diffeomorphisms between compact submanifolds of euclidean space without boundaries. One of the basic tools in [1] was the observation that Gårdings inequality applies to a classical mixed invariant of two positive definite quadratic forms. We will now show that a direct calculation of the mixed invariant allows us to deduce their inequality from Newton's inequality. C. C. Hsiung has pointed out that this is done by a different method in [2].

Let V be an n-dimensional real vector space, and Hom (V, V) the real vector space of all $n \times n$ matrices with real coefficients. Then for $X, Y \in$ Hom (V, V) we introduce functions $P^i(X, Y)$ for $1 \leq i \leq n-1$ by

$$\det (X + tY) = \det X + tP^1(X, Y) + \cdots + t^{n-1}P^{n-1}(X, Y) + t^n \det Y .$$

In particular

$$P^1(X, Y) = \frac{d}{dt} \det (X + tY)|_{t=0} = \langle [X + tY], d(\det) \rangle (X) ,$$

where $[X + tY]$ is the tangent vector to the curve $X + tY$ in $\operatorname{Hom}(V, V)$, and $\langle \ , \ \rangle$ is the canonical bilinear pairing between the tangent and cotangent spaces of $\operatorname{Hom}(V, V)$ at X.

If we introduce the natural coordinates

$$\pi_{ij} : \operatorname{Hom}(V, V) \to R$$

defined for $X = (X_{lm})$ by $\pi_{ij}(X) = X_{ij}$, then

$$\begin{aligned} d(\det)|_X &= \Sigma \frac{\partial \det X}{\partial \pi_{ij}} d\pi_{ij}|_X \\ &= \operatorname{trace} (\operatorname{cofactor} X \cdot dX) , \end{aligned}$$

and

$$\begin{aligned} \langle [X + tY], dX \rangle &= \frac{d}{dt} \pi_{ij}(X + tY)|_{t=0} \\ &= (\pi_{ij}(Y)) = Y . \end{aligned}$$

Therefore by linearity

$$p^1(X, Y) = \operatorname{trace} (\operatorname{cofactor} X \cdot Y) .$$

If X is non-singular, then

$$\operatorname{cofactor} X = (\det X) X^{-1} ,$$

and hence the classical mixed invariant of the pair X, Y utilized by Chern and Hsiung in [1] is

(6.1) $$Y_X = \frac{P^1(X, Y)}{n \det X} = \frac{1}{n} \operatorname{trace} (X^{-1} \cdot Y) .$$

The basic inequality used in [1] is thus equivalent to the fact that positive definite symmetric matricies X, Y satisfy

$$\frac{1}{n} \operatorname{trace} (X^{-1} \cdot Y) \geq \left(\frac{\det Y}{\det X} \right)^{1/n}$$

with equality if and only if Y is congruent by an orthogonal matrix to a multiple of X. By diagonalizing Y with respect to X this is an immediate consequence of Newton's inequality.

Utilizing the explicit expression (6.1) of the mixed invariant, Donald Singley has proved that the integral formula in [1] may be generalized to immersions of compact riemannian manifolds without boundary by the integral formula

$$0 = \int_M d * *_\#^{-1} * \Omega \ .$$

References

[1] S. S. Chern & C. C. Hsiung, *On the isometry of compact submanifolds in Euclidean space*, Math. Ann. **149** (1963) 278–285.

[2] B. H. Rhodes, *On some inequalities of Gårding*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **52** (1966) 594–599.

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